Exponential Separation of Quantum and Classical One-Way Numbers-on-Forehead Communication

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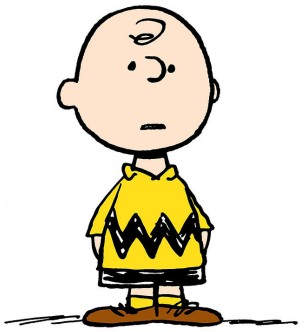
One-Way Number-on-Forehead Communication

$F: [N]^3 \to \{0,1\}$

$\pi_A(y,z)$

$\pi_A(y,z)$

$\pi_B(x,z)$



$y, z \in [N]\times[N]$

Alice

$x, z \in [N]\times[N]$

Bob

$x, y \in [N]\times[N]$

Charlie

$F(x,y,z)$

$$\mathrm{OCC}^{\mathrm{NOF}}(\mathrm{F}) = \min_{\Pi}\{\,|\Pi|:\ \Pi(x,y,z) = F(x,y,z)\ \ \forall\, x,y,z\}$$

Why we study one-way NOF

Circuit Lower Bounds:

[BT94,HG91]: Proving ω (log n) lower bounds for any function f in k-party deterministic one-way NOF communication when k = log n will imply $f \notin ACC^0$ circuits

[Val77,PRS97] : Proving strong **three**-party deterministic one-way NOF communication lower bounds for some problems will imply the size-depth trade-off of Boolean circuits.

Additive combinatorics [LPS17]: .

One-way NOF communication complexity of high dimensional permutations . ⟷

A variety of well known and thoroughly studied problems in combinatorics: Hales-Jewett theorem, dense Ruzsa-Szemerédi graphs, 3-term AP freeness, corner problem

Applications:

Cryptography [CKGS98, BDFP17], Streaming algorithms [KMPV19,VW07], Distributed computing [DKO14], Space-bounded pseudorandom generators [BNS92, GR14], Oblivious branching program lower bounds [VW07].

One-way NOF remain poorly understood

Open Problem 1 [BDPW10] : Optimal explicit separation between the randomized and deterministic one-way NOF communication

$\mathrm{F}: [N]^k \to \{0,1\}$ The deterministic one-way NOF communication complexity of F is $\Omega(\log N)$, but the randomized one-way NOF communication complexity of F is $O(1)$.

Previous results: $\Omega(\log\log N)$ vs $O(1)$ for k parties [BGG06] and $\Omega\left(\log^{1/2} N\right)$ vs $O(1)$ for three parties [LK25]

Open Problem 2 [GP08] : Exponential separation between the quantum and randomized one-way NOF communication

A search problem S The randomized one-way NOF communication complexity of S is $\Omega(poly(\log N))$, but the quantum one-way NOF communication complexity of S is $O(\log\log N)$.

Open Problem 3 : The $\Omega(n)$ randomized one-way three-party NOF communication complexity of set disjointness

Application: The complexity of counting cycles in the adjacency list streaming model [KMPV19]

Quantum versus Classical Separation

Quantum versus classical separation is a central goal in understanding the potential advantages of quantum computation.

Previous works only for two party communication complexity [BCW98, Raz99, BYJK04, GKK+07, RK11, Gav16, GRT22, Gav19, Gav20, GGJL24]

The randomized communication complexity of F is $\Omega(poly(n))$,
but the quantum communication complexity of F is $O(\log n)$.

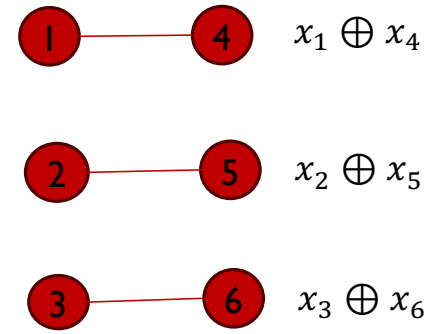
An important open problem [JJGL24][GP08] : Explicit separation between the randomized and quantum NOF communication

Main Theorem: The randomized one-way NOF communication complexity of F is $\Omega(n^{1/3})$,
but the quantum one-way NOF communication complexity of F is $O(\log n)$.

Quantum Advantage in One-way Communication

Hidden Matching Problem(HM)

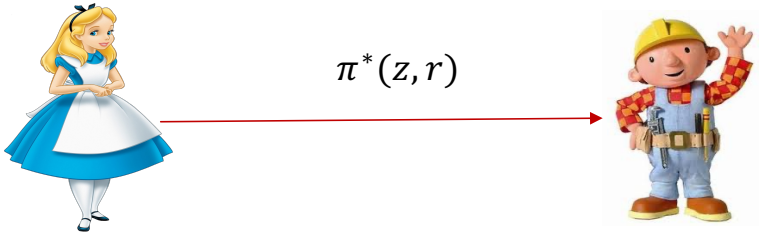
Alice holds $x \in \{0,1\}^n$, Bob holds $k \in [n]$, Bob output a (i, j, b) such that (i, j) is an edge in M_k and $b = x_i \oplus x_j$.



M_k is a perfect matching obtained by cyclically shifting the indices of the second partition by k.

$$M_k := \left\{\left(\ell, m + \left((k + \ell) \bmod m\right)\right) : \ell \in [m]\right\}.$$

Where $m = n/2$



Alice

$x \in \{0,1\}^n$

Bob

M_k

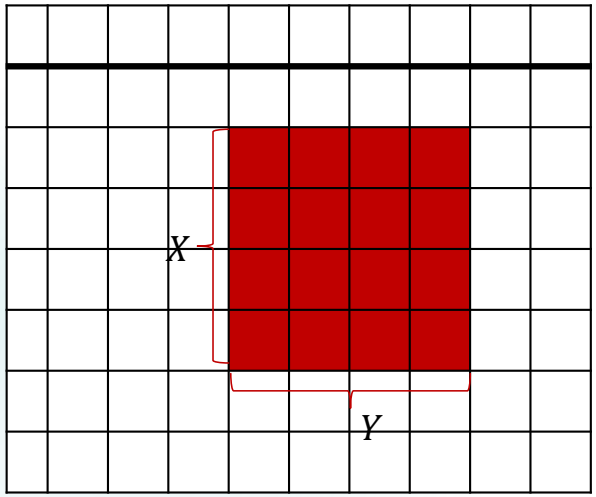
(i, j, b)

Theorem 1 [BYJK04] :
The randomized one-way communication complexity of HM is $\Omega(n^{1/2})$, but the quantum one-way communication complexity of HM is $O(\log n)$.

Can we prove the optimal separation between the randomized and quantum one-way NOF communication via HM ?

IP is a two-source extractor

The communication matrix of $\mathrm{IP}: [N]\times[N] \to [q]$



$\{\mathrm{IP}(x,y): x \in X, y \in Y\} = [q]$

Definition 2

Let q be a prime power and $k \geq 5$. we define the gadget function g is the inner-product function IP : $F_q^k \times F_q^k \to F_q$ given by

$$\mathrm{IP} = \langle\ x, y\ \rangle\ = \sum_{i=1}^{k} x_i y_i\ mod\ \ q$$

By standard fourier analysis

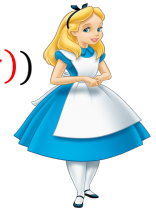
Two Source Extractor Lemma

Set $N\ =\ q^k$ for some constant $k \geq\ 5$, then for any $X, Y \subset\ [N]$ with size $|X|\times\ |Y| \geq \frac{N^2}{q}$,

$$\{\mathrm{IP}(x,y): x \in X, y \in Y\} = [q]$$

Hidden Matching in One-Way NOF

$F(k, y, z) = HM(k, \mathrm{IP}(y, z))$



Alice

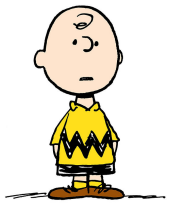
$y, z \in \{0,1\}^n$

$\Pi_A(y, z, r)$

Bob

M_k, z

$\Pi_B(k, z, r)$



Charlie

M_k, y

Lifted Hidden Matching Problem (LHM)

Let $n_0 = n^{2/3}$, $\mathrm{IP}: \{0,1\}^n \times \{0,1\}^n \to \{0,1\}^{n_0}$, $k \in [n_0]$.

Alice holds $y, z \in \{0,1\}^n$, Bob holds $k \in [n_0], z \in \{0,1\}^n$, Charlie holds $k \in [n_0], y \in \{0,1\}^n$.

Charlie outputs (i, j, b) such that (i, j) is an edge in M_k and $b = \mathrm{IP}(y, z)_i \oplus \mathrm{IP}(y, z)_j$

The quantum one-way NOF communication complexity of LHM is $O(\log n)$.

The randomized one-way NOF communication complexity of LHM is $\Omega(n^{1/3})$.

Step 1: Simplifying the protocols

$\mathsf{F}(k, y, z) = HM(\mathrm{IP}(y, z), k)$

$y, z \in \{0,1\}^n$

Alice

$\Pi = (\Pi_A, \Pi_B)$

$\Pi_A(y, z, r)$

$\Pi_B(M_k, z, r)$

Bob

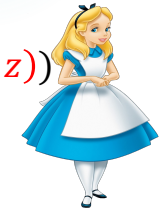
M_k, z

Charlie

M_k, y

$|\Pi_A^*| = |\Pi_A|$ and $|\Pi_B^*| = n_0 \cdot |\Pi_B|$

$\mathsf{F}(M, y, z) = HM(k, \mathrm{IP}(y, z))$



$y, z \in \{0,1\}^n$

Alice

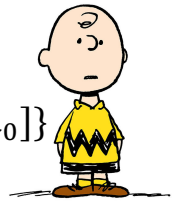
$\Pi^* = (\Pi_A^*, \Pi_B^*)$

$\Pi_A^*(y, z, r)$

$\Pi_B^*(z, r) = \{\Pi_B(M_k, z, r) : k \in [n_0]\}$

Bob

z



Charlie

M_k, y

Step 2: Information from Alice

Alice

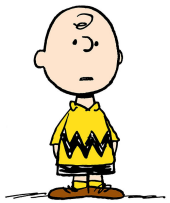
$y, z \in \{0,1\}^n$

$\Pi_A^*(y, z, r)$

Bob

z

$\Pi_B^*(z, r)$



Charlie

M_k, y

Case 1:

Alice's message contains sufficient information about $\mathrm{IP}(y, z)$ such that Charlie can solve $\mathrm{HM}(\mathrm{k}, \mathrm{IP}(\mathrm{y}, \mathrm{z}))$.

By the randomized one-way communication complexity of HM ,

$$|\Pi_A| = |\Pi_A^*| = \Omega(\sqrt{n_0}) = \Omega(n^{1/3})$$

$\mathrm{LHM}(\mathrm{k}, \mathrm{y}, \mathrm{z}) = \mathrm{HM}(\mathrm{k}, \mathrm{IP}(\mathrm{y}, \mathrm{z}))$ $\quad \mathrm{IP}: \{0,1\}^n \times \{0,1\}^n \to \{0,1\}^{n_0}$

Step 3: Information form Bob

Alice

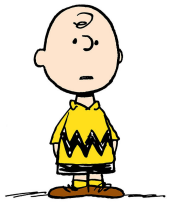
$y, z \in \{0,1\}^n$

$\Pi_A^*(y, z, r)$

Bob

z

$\Pi_B^*(z, r)$



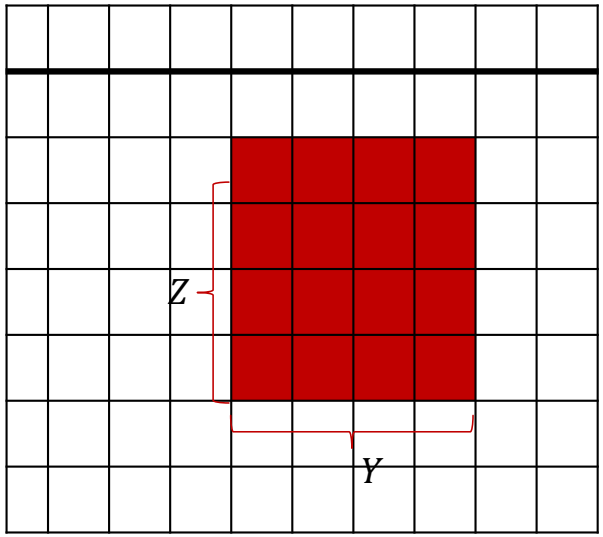
Charlie

M, y

$\mathrm{LHM}(\mathrm{k}, \mathrm{y}, \mathrm{z}) = \mathrm{HM}(\mathrm{k}, \mathrm{IP}(\mathrm{y}, \mathrm{z}))$

Case 2:

Bob sends sufficient information about z through one-way communication such that Charlie can solve $\mathrm{HM}(\mathrm{k}, \mathrm{IP}(\mathrm{y}, \mathrm{z}))$.



$\{\mathrm{IP}(y, z): y \in Y, z \in Z\} = [q]$

Step 3: Information form Bob

Alice

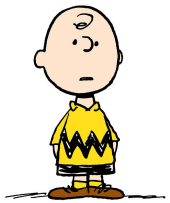
$y, z \in \{0,1\}^n$

$\Pi_A^*(y, z, r)$

Bob

z

$\Pi_B^*(z, r)$



Charlie

M_k, y

$\mathrm{LHM}(\mathrm{k}, \mathrm{y}, \mathrm{z}) = \mathrm{HM}(\mathrm{k}, \mathrm{IP}(\mathrm{y}, \mathrm{z}))$

Case 2:

Bob sends sufficient information about z through one-way communication such that Charlie can solve $\mathrm{HM}(\mathrm{k}, \mathrm{IP}(\mathrm{y}, \mathrm{z}))$.

By the two-source extractor properties of IP,

$$|\Pi_B^*| = \Omega(n)$$

By the properties of Π_B^* ,

$$|\Pi_B| = \Omega\left(\frac{\Pi_B^*}{n_0}\right) = \Omega(n^{1/3})$$

Finishing the proof

Alice

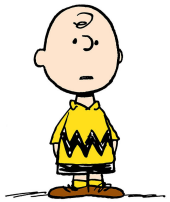
$y, z \in \{0,1\}^n$

$\Pi_A^*(y, z, r)$

Bob

z

$\Pi_B^*(z, r)$



Charlie

M_k, y

$\mathrm{LHM}(k, y, z) = \mathrm{HM}(k, \mathrm{IP}(y, z))$

One of Case 1 and Case 2 must be true:

$\mathrm{ORCC}^{\mathrm{NOF}}(\mathrm{LHM}) = |\Pi_A| + |\Pi_B| = \Omega(n^{1/3})$

Open Problem

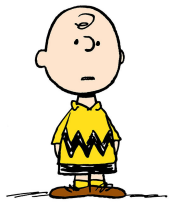
$F(x, y, z) = f(x, \mathrm{IP}(y, z))$

Alice

$y, z \in \{0,1\}^n$

$\pi_A(y, z)$

$\pi_A(y, z)$



$\pi_B(x, z)$

Bob

$x, z \in \{0,1\}^n$

Charlie

$x, y \in \{0,1\}^n$

Conjecture:

For any Boolean function f : $[q] \times [q] \to \{0,1\}$, we have

$$\mathrm{OCC}^{\mathrm{NOF}}(f * \mathrm{IP}) = \Theta(\mathrm{ODCC}(f))$$

Applications:

[LPS17] Optimal explicit separation between the randomized and deterministic NOF communication (one of the major open problems in NOF communication)

[Val77] Size-depth trade-off of Boolean circuits (a long-standing open problem in circuit complexity)

Quantum Upper bound

[BYJK04]: The quantum one-way communication complexity of HM is $O(\log n)$.

Alice sends the state

$$|\psi\rangle = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (-1)^{z_i} |i\rangle$$

Bob performs a measurement on the state $|\psi\rangle$ in the orthonormal basis

$$B = \{\frac{1}{\sqrt{2}}(|k\rangle \pm |l\rangle) | (k,l) \in M\}.$$

The probability that the outcome of the measurement is a basis state $\frac{1}{\sqrt{2}}(|k\rangle \pm |l\rangle$ is

$$\frac{1}{\sqrt{2}}(|k\rangle + |l\rangle : \quad \left|\left\langle \psi \middle| \frac{1}{\sqrt{2}}(|k\rangle + |l\rangle)\right\rangle\right|^2 = \frac{1}{2n}\left((-1)^{x_k} + (-1)^{x_\ell}\right)^2 \qquad \frac{2}{n} \; if \; x_k \oplus x_\ell = 0 \; and \; 0 \; otherwise$$

$$\frac{1}{\sqrt{2}}(|k\rangle - |l\rangle) : \quad \left|\left\langle \psi \middle| \frac{1}{\sqrt{2}}(|k\rangle - |l\rangle)\right\rangle\right|^2 = \frac{1}{2n}\left((-1)^{x_k} - (-1)^{x_\ell}\right)^2 \qquad \frac{2}{n} \; if \; x_k \oplus x_\ell = 1 \; and \; 0 \; otherwise$$